UNIQUELY D-COLOURABLE DIGRAPHS WITH LARGE GIRTH

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Abstract. Let C and D be digraphs. A mapping $f : V(D) \to V(C)$ is a C-colouring if for every arc uv of D, either $f(u)f(v)$ is an arc of C or $f(u) = f(v)$, and the preimage of every vertex of C induces an acyclic subdigraph in D. We say that D is C-colourable if it admits a C-colouring and that D is uniquely C-colourable if it is surjectively C-colourable and any two C-colourings of D differ by an automorphism of C. We prove that if a digraph D is not C-colourable, then there exist digraphs of arbitrarily large girth that are D-colourable but not C-colourable. Moreover, for every digraph D that is uniquely D-colourable, there exists a uniquely D-colourable digraph of arbitrarily large girth. In particular, this implies that for every rational number $r \geq 1$, there are uniquely circularly r-colourable digraphs with arbitrarily large girth.

1. Introduction

In a seminal *Canadian Journal of Mathematics* article [12], Paul Erdős established nonconstructively the existence of graphs with arbitrarily large girth γ and arbitrarily large chromatic number χ. In these introductory remarks, we focus mainly on Erdős' theorem, for the features that make it interesting are shared by its progeny (e.g. [6, 28, 31]), the first of which also appeared in *CJM*.

Because graphs with similar combined properties as guaranteed by Erdős had been constructed earlier—e.g. triangle-free plus large χ [8, 24, 34] or girth at least six plus large χ [9, 16]—his result was not wholly unanticipated. Nevertheless, it remains somewhat counterintuitive. Naïvely, one might reason that having large girth implies edge-sparsity, while having large chromatic number entails edge-abundance, so how could both properties coexist in one graph? Even upon closer inspection, the result seems paradoxical. If, for a positive integer ℓ, a graph G satisfies $\gamma > \ell$, then any set of at most ℓ vertices induces an acyclic, hence 2-colourable, subgraph.

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Why is it not possible to assemble such colourings into a proper colouring of $V(G)$ using few colours? These questions' answers might be regarded as the take-home message of Erdős' theorem: since the chromatic number depends intrinsically upon the graph's global structure, local 2-colourability imparts nothing on χ.

Yet the theorem's influence somehow manages to transcend its important message. Every student of combinatorial probability studies Erdős' proof (cf. [1, 2, 5, 7, 10, 22]), that employs a deterministic step after a probabilistic argument for the existence of graphs with *few* short cycles and small stability number. And though constructive proofs [18, 20, 26] of Erdős' existence theorem eventually followed, their complexity perhaps precludes their inclusion in 'The Book', generally imagined to favour the elegance, clarity, and simplicity of Erdős' original argument.

Aside from its beautiful proof, the theorem's influence can also be measured by considering its descendants. Nešetřil [25] conjectured, and Bollobás and Sauer [6] proved, the existence of graphs as guaranteed by Erdős that are, moreover, uniquely χ-colourable. Colourings are special cases of homomorphisms into a fixed graph, and Zhu [31] extended both Erdős' and Bollobás and Sauer's results to homomorphisms into general graphs. Rather recently, the results of [31] were extended by Nešetřil and Zhu [28] to give a simultaneous generalization of Zhu's two primary results. Without attempting to give an exhaustive list, we also note the appearance in recent years of a host of other articles related to the interplay between girth and colouring; see, e.g., [11, 17, 19, 23, 27, 29, 32]. The results of the present paper extend the main theorems of Zhu [31] to digraphs with acyclic homomorphisms.

Notation, terminology, details. As much as possible, we try to follow standard terminology. See, for example, [3, 7] for graphs and digraphs, [2, 22] for probabilistic concerns, and [15] for homomorphisms.

Our digraphs are simple—i.e. loopless and without multiple arcs—however, we allow two vertices u, v to be joined by two oppositely directed arcs, uv and vu. The *girth* of a graph or digraph refers to the length of a shortest cycle, that we take to mean directed cycle in the digraph case (and infinite in either acyclic case).

Recall that a *homomorphism* of a graph G into a graph H is a function $\phi\colon V(G) \to V(H)$ such that $\{\phi(u), \phi(v)\} \in E(H)$ whenever $\{u, v\} \in E(G)$. An *acyclic homomorphism* of a digraph D into a digraph C is a function $\phi\colon V(D) \to V(C)$ such that:

(i) for every vertex $v \in V(C)$, the subdigraph of D induced by $\phi^{-1}(v)$ is acyclic;
(ii) for every arc $uv \in E(D)$, either $\phi(u) = \phi(v)$, or $\phi(u)\phi(v)$ is an arc of C.

If digraphs D and C are obtained from undirected graphs G and H, respectively, by replacing every edge by two oppositely directed arcs, then acyclic homomorphisms

between D and C correspond to usual graph homomorphisms between G and H. In this sense, acyclic homomorphisms can be viewed as a generalization of the notion of homomorphisms of undirected graphs.

It is well-known and easy to see that a graph G is (properly) r-colourable (for a positive integer r) if and only if G admits a homomorphism to the complete graph K_r. Thus, G is commonly called *H-colourable* if there is a homomorphism from G to H. In the same way as homomorphisms generalize the notion of graph colouring, acyclic homomorphisms generalize digraph colouring; cf. [4]. Motivated by this, we say that a digraph D is *C-colourable* if there is an acyclic homomorphism from D to C.

Zhu generalized Erdős' theorem as follows.

Theorem 1.1 ([31]). *If G and H are graphs such that G is not H-colourable, then for every positive integer g, there exists a graph G^* of girth at least g that is G-colourable but not H-colourable.*

To recover Erdős' theorem, suppose that we want to arrange for $\gamma \geq g$ and $\chi \geq r$, for some prescribed integers g and r; then we take $G = K_r$ and $H = K_{r-1}$ in Theorem 1.1.

Our first main result is a digraph analogue of the preceding result.

Theorem 1.2. *If D and C are digraphs such that D is not C-colourable, then for every positive integer g, there exists a digraph D^* of girth at least g that is D-colourable but not C-colourable.*

Just as Theorem 1.1 generalizes Erdős' theorem, Theorem 1.2 generalizes the analogue appearing in [4]. See the introduction to Section 4 for a statement of this analogue.

A graph G is *uniquely H-colourable* if it is surjectively H-colourable, and for any two H-colourings ϕ, ψ of G, there is an automorphism π of H such that

$$\phi = \pi \circ \psi. \tag{1.1}$$

Unique D-colourability is defined analogously for digraphs D. In either case, when (1.1) occurs, we sometimes say that ϕ and ψ *differ by an automorphism* of H. A graph H is a *core* if it is uniquely H-colourable; likewise for digraphs. To align this formulation with the usual one (cf. [13, 15]), we offer the following observation about the digraph version.

Lemma 1.3. *A digraph D is a core if and only if every acyclic homomorphism $V(D) \to V(D)$ is a bijection.*

Proof. Let $\phi : V(D) \to V(D)$ be an acyclic homomorphism. If ϕ is not a bijection, then ϕ and the identity homomorphism do not differ by an automorphism of D, so D is not a core.

Suppose now that D is not a core, and let ϕ, ψ be two acyclic homomorphisms that do not differ by an automorphism of D. If ϕ (or ψ) is bijective, then it is a homomorphism of D onto itself. This implies that it is an automorphism of D. Therefore, ϕ and ψ are not both bijective. □

Zhu generalized the aforementioned Bollobás-Sauer theorem [6] as follows.

Theorem 1.4 ([31]). *For every graph H that is a core and every positive integer g, there exists a graph H^* of girth at least g that is uniquely H-colourable.*

Bollobás and Sauer's result follows from Theorem 1.4 because complete graphs are cores, as is easily verified.

Our second main result establishes a digraph analogue of Theorem 1.4.

Theorem 1.5. *For every digraph D that is a core and every positive integer g, there exists a digraph D^* of girth at least g that is uniquely D-colourable.*

Theorem 1.5 immediately applies to digraph colourings and digraph circular colourings (see, e.g., [4, 21]) to yield our third main result, Theorem 4.4. In favour of an abbreviated mention of this result here, we postpone until Section 4 its full statement, the definition of 'circular colouring', and some related discussion.

Corollary 1.6. *For every rational number $r \geq 1$ and every positive integer g, there exists a digraph of girth at least g that is uniquely circularly r-colourable.*

We devote Section 2 to the proof of Theorem 1.2, while Section 3 contains the proof of Theorem 1.5. Both proofs are probabilistic and follow the main ideas of [6] and [31], which themselves trace back to Erdős [12]. However, just as both of these earlier refinements required new ideas to move to the next level, additional care and some inspiration are needed to extend the proofs to the digraph setting.

2. Proof of Theorem 1.2

We begin by setting up a suitable random digraph model. Suppose that $V(D) = \{1, 2, \ldots, k\}$ and that $q = |E(D)|$. Let n be a (large) positive integer, and let D_n be the digraph obtained from D as follows: replace every vertex i with a (temporarily) stable set V_i of n ordered vertices $v_1, v_2, ..., v_n$, and replace each arc ij of D by the set of all possible n^2 arcs from V_i to V_j; additionally, add each arc $v_r v_s$ such that $v_r, v_s \in V_i$ and $r < s$. Clearly, $|V(D_n)| = kn$ and $|E(D_n)| = qn^2 + k\binom{n}{2}$.

Now fix a positive $\varepsilon < 1/(4g)$. Our random digraph model $\mathcal{D} = \mathcal{D}(D_n, p)$ consists of those spanning subdigraphs of D_n in which the arcs of D_n are chosen randomly and independently with probability $p = n^{\varepsilon - 1}$.

As usual in nonconstructive probabilistic proofs of results of this nature (cf. [6, 28, 31]), the idea is to show that most digraphs in $\mathcal{D}$ have only a few short cycles, and for most digraphs $H \in \mathcal{D}$, the subdigraph of H obtained by removing an arbitrary yet small set of arcs is not C-colourable. Choosing an $H \in \mathcal{D}$ with both these properties, we can force the girth to be large by deleting an arc from each short cycle. Since the set A_0 of deleted arcs is small, the resulting digraph $H - A_0$ satisfies the desired conclusion of Theorem 1.2.

To make this description more precise, let $\mathcal{D}_1$ denote the set of digraphs in $\mathcal{D}$ containing at most $\lceil n^{g\varepsilon} \rceil$ cycles of length less than g, and let $\mathcal{D}_2$ be the set of digraphs $H \in \mathcal{D}$ that have the property that $H - A_0$ is not C-colourable for any set A_0 of at most $\lceil n^{g\varepsilon} \rceil$ arcs. We will show that

$$|\mathcal{D}_1| > \left(1 - n^{-\varepsilon/2}\right) |\mathcal{D}| \tag{2.1}$$

and

$$|\mathcal{D}_2| > \left(1 - e^{-n}\right) |\mathcal{D}| . \tag{2.2}$$

Since (2.1) and (2.2) imply that $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \varnothing$ (for sufficiently large n), there exists a digraph $H \in \mathcal{D}_1 \cap \mathcal{D}_2$. Now $H \in \mathcal{D}_1$ implies that there is a set A_0 of at most $\lceil n^{g\varepsilon} \rceil$ arcs whose removal leaves a digraph $D^* := H - A_0$ of girth at least g, while $H \in \mathcal{D}_2$ means that D^* is not C-colourable. Thus, it remains to establish (2.1) and (2.2).

Proof of (2.1). The expected number N_ℓ of cycles of length ℓ in a digraph $H \in \mathcal{D}$ is at most

$$\binom{kn}{\ell} (\ell - 1)!\, p^\ell \tag{2.3}$$

since there are $\binom{kn}{\ell}(\ell - 1)!$ ways of choosing a cyclic sequence of ℓ vertices as a candidate for a cycle, and such an ℓ-cycle occurs in $\mathcal{D}$ with probability either 0 or p^ℓ. It is easy to see that the product of the first two factors in (2.3) is smaller than $(kn)^\ell/\ell$. Therefore, if n is large enough, then

$$\sum_{\ell=2}^{g-1} N_\ell \leq \sum_{\ell=2}^{g-1} \frac{(kn^\varepsilon)^\ell}{\ell} < k^{g-1} n^{(g-1)\varepsilon} < n^{-\varepsilon/2} n^{g\varepsilon}.$$

Now (2.1) follows easily from Markov's Inequality. □

Proof of (2.2). We shall argue that $|\mathcal{D} \setminus \mathcal{D}_2| < e^{-n} |\mathcal{D}|$. If $H \in \mathcal{D} \setminus \mathcal{D}_2$, then there is a set A_0 of at most $\lceil n^{g\varepsilon} \rceil$ arcs of H so that $H - A_0$ admits an acyclic homomorphism h to C. Let $k' = |V(C)|$. By the pigeonhole principle, for each $i \in V(D)$, there exists a vertex $x_i \in V(C)$ such that $|V_i \cap h^{-1}(x_i)| \geq n/k'$. Define $\phi \colon V(D) \to V(C)$ by setting $\phi(i) = x_i$. Since $n/k' \gg n^{g\varepsilon}$, the set $V_i \cap h^{-1}(x_i)$ contains a subset W_i of cardinality $w := \lceil n/(2k') \rceil$ such that no arc in A_0 has an end vertex in W_i.

Since D is not C-colourable, the function ϕ is not an acyclic homomorphism. Therefore, either there is an arc $ij \in E(D)$ such that $\phi(i) \neq \phi(j)$ and $\phi(i)\phi(j)$ is not an arc of C, or there is a vertex $v \in V(C)$ such that the subdigraph of D induced on $\phi^{-1}(v)$ contains a cycle.

We first consider the case when ij is an arc of D such that $\phi(i) \neq \phi(j)$ and $\phi(i)\phi(j)$ is not an arc of C. Since h is an acyclic homomorphism, there are no arcs from W_i to W_j in $H - A_0$. By the definition of W_i and W_j, neither are there such arcs in H.

Let us now estimate the expected number M of pairs of sets $A \subseteq V_i$, $B \subseteq V_j$, with $|A| = |B| = w$, such that $ij \in E(D)$ and such that there is no arc from A to B in $H \in \mathcal{D}$ (we call such a pair A, B a *bad pair*). By the linearity of expectation, we have

$$M = q\binom{n}{w}^2 (1-p)^{w^2} < q\left(\frac{n^w}{w!}\right)^2 (1-p)^{w^2} = \frac{q(n^2(1-p)^w)^w}{(w!)^2}. \tag{2.4}$$

Since w grows no more (or less) than linearly with n, for sufficiently large n we have

$$n^2(1-p)^w < e^{-2k'} \text{ and } \frac{q}{(w!)^2} < \frac{1}{2}.$$

Therefore, Markov's Inequality and (2.4) yield

$$\Pr(\exists \text{ a bad pair}) < \frac{e^{-n}}{2}. \tag{2.5}$$

Suppose now that there is a vertex $v \in V(C)$ such that D contains a cycle Q whose vertices are all in $\phi^{-1}(v)$. Suppose that $Q = i_1 i_2 \cdots i_t$. Observe that $2 \leq t \leq k$. Since $\phi(Q) = \{v\}$, we conclude that $h(W_{i_1}) = h(W_{i_2}) = \cdots = h(W_{i_t}) = \{v\}$. Since h is an acyclic homomorphism, the subdigraph of H induced on $W_{i_1} \cup W_{i_2} \cup \cdots \cup W_{i_t}$ is acyclic.

Let us consider all sequences of sets $U_{j_1}, U_{j_2}, \ldots, U_{j_\ell}$ such that, for $r = 1, 2, \ldots, \ell$, we have $U_{j_r} \subseteq V_{j_r}$ and $|U_{j_r}| = w$, and the vertex sequence $j_1 j_2 \cdots j_\ell$ is a cycle in D. Let $U(\ell)$ denote the subdigraph of H induced on $U_{j_1} \cup U_{j_2} \cup \cdots \cup U_{j_\ell}$, and let $P_\ell := \Pr(U(\ell) \text{ is acyclic})$. We call this sequence of sets *bad* if $U(\ell)$ is acyclic. Since the expected number N of bad sequences is the sum of the corresponding expectations over all possible cycle lengths, we have

$$N \leq \sum_{\ell=2}^{k} \binom{k}{\ell} (\ell-1)! \binom{n}{w}^{\ell} P_\ell. \tag{2.6}$$

In order to bound N, we first bound the probabilities P_ℓ.

Lemma 2.1. *There exists a constant $\gamma > 0$ (not depending on n) such that $P_\ell \leq e^{-\gamma n^{1+\varepsilon}}$ for every integer $\ell \in \{2, 3, \ldots, k\}$.*

We present two proofs of Lemma 2.1. The second invokes the Janson Inequalities (see, e.g., [2, Chapter 8]). The first uses only elementary methods and relies in the beginning on the following observation.

Lemma 2.2. *A digraph D is acyclic if and only if every induced subdigraph contains a vertex of outdegree* 0*.*

Proof. If D is acyclic, then every induced subdigraph of D must be acyclic and therefore must contain a vertex of outdegree 0. If D is not acyclic, then it must contain a cycle, the vertex set of which induces a subdigraph containing no vertex of outdegree 0. $\square$

Proof 1 of Lemma 2.1. Let E_0 be certain ($\Pr(E_0) = 1$), and let E_j be the event that all induced subdigraphs of $U(\ell)$ with more than $\ell w - j$ vertices have a vertex of outdegree 0 (the outdegree in the induced subdigraph). Lemma 2.2 shows that

$$P_\ell = \Pr\Big(\bigcap_{j=0}^{\ell w} E_j\Big) = \prod_{j=0}^{\ell w-1} \Pr(E_{j+1}|E_j) \le \prod_{j=0}^{w-1} \Pr(E_{j+1}|E_j). \tag{2.7}$$

We will call a set $S \subseteq V(U(\ell))$ an *acyclic-sink set* if the induced subdigraph $U(\ell)[S]$ is acyclic and there are no arcs in $U(\ell)$ from S to $V(U(\ell)) \setminus S$ (so S acts as a sink in $U(\ell)$).

Claim 1: The union of two acyclic-sink sets in $U(\ell)$ is an acyclic-sink set in $U(\ell)$.

Proof of claim. Let A and B be two acyclic-sink sets in a digraph $U(\ell)$. Since A and B are both sinks in $U(\ell)$, their union $A \cup B$ is a sink because there are no arcs from $A \cup B$ to $V(U(\ell)) \setminus (A \cup B)$. Consider the three sets $A \setminus B$, $B \setminus A$, and $A \cap B$; each is a subset of an acyclic-sink set so each induces an acyclic digraph. Since A is a sink in $U(\ell)$, there can be no arcs from $A \cap B$ to $B \setminus A$. Likewise B is a sink in $U(\ell)$, so there can be no arcs from $A \cap B$ to $A \setminus B$. Therefore, $A \cup B$ induces an acyclic digraph and is consequently an acyclic-sink set in $U(\ell)$. $\square$

Claim 2: There exists an acyclic-sink set $S \subseteq V(U(\ell))$ of cardinality j if and only if E_j occurs.

Proof of claim. If there exists an acyclic-sink set of cardinality j, then a subdigraph of $U(\ell)$ with more than $\ell w - j$ vertices must have a nonempty intersection with it. Any subdigraph that has nonempty intersection with an acyclic-sink set induces a subdigraph containing a vertex of outdegree zero.

If there is no acyclic-sink set of cardinality j, then the largest acyclic-sink set is an $S' \subseteq V(U(\ell))$ such that $|S'| < j$. Then $U(\ell) - S'$ is a subdigraph of $U(\ell)$ with cardinality greater than $\ell w - j$ and with no vertices of outdegree 0 (otherwise we could have added them to S' and had a larger acyclic-sink set). $\square$

Claim 3: If $U(\ell)$ has an acyclic-sink set of cardinality j, then it has an acyclic-sink set of cardinality $j-1$.

Proof of claim. Suppose that S is an acyclic-sink set in $U(\ell)$ of cardinality j. Then the subdigraph $U(\ell)[S]$ is acyclic, so there must be a vertex v with indegree 0 in $U(\ell)[S]$. Consider the set $S \setminus \{v\}$; this induces an acyclic subdigraph of $U(\ell)$ because it is a subdigraph of an acyclic digraph. There were no arcs from S to $V(U(\ell)) \setminus S$, and there are no arcs from $S \setminus \{v\}$ to v, so $S \setminus \{v\}$ is a sink in $U(\ell)$. Therefore, there exists an acyclic-sink set in $U(\ell)$ of cardinality $j-1$. □

We now fix j in order to estimate $\Pr(E_{j+1}|E_j)$. Let $I = \{1, 2, \ldots, \binom{\ell w}{j}\}$ and let $\{S_i\}_{i \in I}$ be the j-subsets of the ℓw vertices of $U(\ell)$ (in some fixed order). Let B_i be the event that S_i is an acyclic-sink set in $U(\ell)$. By Claim 1, if more than one B_i occurs, there must be an acyclic-sink set of cardinality at least $j+1$, and so by Claim 3, there exists one of cardinality exactly $j+1$. Therefore by Claim 2,

$$\Pr\left(E_{j+1}|\bigcap_{i \in Y} B_i\right) = 1 \text{ whenever } Y \subseteq I \text{ and } |Y| \geq 2. \tag{2.8}$$

Now additionally fix a B_i, and we will estimate $\Pr(E_{j+1}|B_i)$. Let F be the event that $U(\ell) - S_i$ contains a vertex of outdegree 0. Then

$$\Pr(E_{j+1}|B_i) = \Pr(E_{j+1}|F \cap B_i)\Pr(F|B_i) + \Pr(E_{j+1}|F^C \cap B_i)\Pr(F^C|B_i). \tag{2.9}$$

The event E_{j+1} occurs when all subsets of $V(U(\ell))$ of cardinality greater than $\ell w - (j+1)$ induce a subdigraph in $U(\ell)$ that has a vertex of outdegree 0. Clearly $U(\ell) - S_i$ has cardinality $\ell w - j$, while F^C is the event that this set induces a subdigraph with no vertex of outdegree zero. Thus $\Pr(E_{j+1}|F^C \cap B_i) = 0$. All sets of cardinality exceeding $\ell w - (j+1)$ that are distinct from $V(U(\ell)) \setminus S_i$ have a nonempty intersection with S_i, which (given B_i) is an acyclic-sink set in $U(\ell)$. Therefore, subdigraphs of $U(\ell)$ induced on these sets have a vertex of outdegree 0, so that $\Pr(E_{j+1}|B_i \cap F) = 1$. Using these observations, (2.9) reduces to $\Pr(E_{j+1}|B_i) = \Pr(F|B_i)$. The event F is independent of the event B_i since the vertices in S_i do not affect the outdegree of vertices in the subdigraph induced by $V(U(\ell)) \setminus S_i$. Therefore, $\Pr(E_{j+1}|B_i) = \Pr(F)$.

Now we estimate the probability of F. The probability that any particular vertex of $U(\ell) - S_i$ has outdegree 0 in the induced subdigraph is bounded from above by $(1-p)^{(w-j)}$. Since these outdegree computations are independent for each vertex, the probability that all vertices have outdegree greater than 0 is bounded from below by $(1-(1-p)^{(w-j)})^{(\ell w - j)}$, so that

$$\begin{aligned} \Pr(E_{j+1}|B_i) = \Pr(F) &\leq 1 - ((1-(1-p)^{(w-j)})^{(\ell w - j)}) \\ &< (\ell w - j)(1-p)^{(w-j)} =: p_j. \end{aligned} \tag{2.10}$$

We also need to estimate $\Pr(E_{j+1}|E_j)$. By Claim 2, E_j occurs if and only if $\bigcup_{i\in I} B_i$ occurs. Thus we may rewrite $\Pr(E_{j+1}|E_j)$ using inclusion-exclusion:

$$\begin{aligned}
\Pr(E_{j+1}|E_j) &= \Pr\Big(E_{j+1}|\bigcup_{i\in I} B_i\Big)\\
&= \frac{\Pr\Big(E_{j+1}\cap\big(\bigcup_{i\in I} B_i\big)\Big)}{\Pr\big(\bigcup_{i\in I} B_i\big)}\\
&= \frac{\Pr\Big(\bigcup_{i\in I}(E_{j+1}\cap B_i)\Big)}{\Pr\big(\bigcup_{i\in I} B_i\big)}\\
&= \sum_{\varnothing\neq Y\subseteq I}(-1)^{|Y|+1}\frac{\Pr\Big(E_{j+1}\cap\big(\bigcap_{y\in Y} B_y\big)\Big)}{\Pr\big(\bigcup_{i\in I} B_i\big)}\\
&= \sum_{\varnothing\neq Y\subseteq I}(-1)^{|Y|+1}\frac{\Pr\Big(E_{j+1}\cap\big(\bigcap_{y\in Y} B_y\big)\Big)}{\Pr\big(\bigcap_{y\in Y} B_y\big)}\frac{\Pr\big(\bigcap_{y\in Y} B_y\big)}{\Pr\big(\bigcup_{i\in I} B_i\big)}\\
&= \sum_{\varnothing\neq Y\subseteq I}(-1)^{|Y|+1}\Pr\Big(E_{j+1}|\bigcap_{y\in Y} B_y\Big)\Pr\Big(\bigcap_{y\in Y} B_y|\bigcup_{i\in I} B_i\Big)\\
&= \sum_{y\in I}\Pr(E_{j+1}|B_y)\Pr\Big(B_y|\bigcup_{i\in I} B_i\Big)\\
&\quad+\sum_{\substack{Y\subseteq I\\ |Y|\geq 2}}(-1)^{|Y|+1}\Pr\Big(E_{j+1}|\bigcap_{y\in Y} B_y\Big)\Pr\Big(\bigcap_{y\in Y} B_y|\bigcup_{i\in I} B_i\Big).
\end{aligned}$$

Using (2.8) and (2.10) in the last expression for $\Pr(E_{j+1}|E_j)$ gives

$$\begin{aligned}
\Pr(E_{j+1}|E_j) &\leq p_j\sum_{y\in I}\Pr\Big(B_y|\bigcup_{i\in I} B_i\Big)+\sum_{\substack{Y\subseteq I\\ |Y|\geq 2}}(-1)^{|Y|+1}\Pr\Big(\bigcap_{y\in Y} B_y|\bigcup_{i\in I} B_i\Big)\\
&= p_j\sum_{y\in I}\Pr\Big(B_y|\bigcup_{i\in I} B_i\Big)+\Big[\Pr\Big(\bigcup_{i\in I} B_i|\bigcup_{i\in I} B_i\Big)-\sum_{y\in I}\Pr\Big(B_y|\bigcup_{i\in I} B_i\Big)\Big]\\
&= p_j\sum_{y\in I}\Pr\Big(B_y|\bigcup_{i\in I} B_i\Big)+\Big[1-\sum_{y\in I}\Pr\Big(B_y|\bigcup_{i\in I} B_i\Big)\Big].
\end{aligned}$$

Since $\sum_{y\in I}\Pr\big(B_y|\bigcup_{i\in I} B_i\big)\geq 1$ and $p_j-1<0$, we have

$$\Pr(E_{j+1}|E_j) \leq 1+\sum_{y\in I}\Pr\Big(B_y|\bigcup_{i\in I} B_i\Big)(p_j-1)<p_j.$$

Applying this last estimate to (2.7) yields

$$\begin{aligned}
P_\ell &\le \prod_{j=0}^{w-1} p_j = \prod_{j=0}^{w-1} (\ell w - j)(1-p)^{(w-j)} \\
&< (\ell w)^w (1-p)^{w(w+1)/2} \\
&\le (\ell w)^w (1-p)^{w^2/2} \\
&\le (\ell w)^w e^{-pw^2/2} \\
&\le \left(\ell w e^{-pw/2}\right)^w \\
&\le \left(\ell w e^{-n^\varepsilon/(4k')}\right)^w \qquad (2.11)\\
&\le \left(e^{-n^\varepsilon/(5k')}\right)^w \qquad (2.12)\\
&\le e^{-n^{1+\varepsilon}/(10(k')^2)}. \qquad (2.13)
\end{aligned}$$

In passing from (2.11) to (2.13), the reader may find it helpful to recall that $n = |V_i|$ (for $1 \le i \le k$), $k' = |V(C)|$, ℓ is between 2 and k, $w = \lceil n/(2k') \rceil$, and $p = n^{\varepsilon-1}$, and that these estimates are valid for fixed k' and sufficiently large n. Of course, Lemma 2.1 follows if we take $\gamma = 1/(10(k')^2)$. $\square$

Proof 2 of Lemma 2.1. We use the Janson Inequalities, (mainly) follow the notation of [2, Chapter 8], and assume familiarity on the readers' part. Here, Ω denotes the set of all potential arcs (in D_n, as defined at the start of Section 2) between the sets U_{j_i}, for $i = 1, 2..., \ell$, (introduced just prior to our statement of Lemma 2.1); each arc in Ω appears with probability p.

Let s be a (large) multiple of ℓ; the value of s will be independent of n and specified below. Now, let us enumerate those cycles of D_n that are of length s, and that cyclically traverse $U_{j_1}, U_{j_2}, ..., U_{j_\ell}$ s/ℓ times. For $j \ge 1$, denote by S_j the arc set of the jth such cycle and by $\mathcal{B}_j$ the event that the arcs in S_j all appear in H (i.e. the cycle determined by S_j is present in H). Let the random variable X count those $\mathcal{B}_j$ that occur. Since $\Pr(X = 0)$ (the probability that there is no such cycle of length s) is an upper bound for P_ℓ (the chance that $U(\ell)$ is acyclic), we can bound P_ℓ by bounding $\Pr(X = 0)$, and estimating the latter quantity is exactly the purpose of Janson's Inequalities. In the Janson paradigm, the value of Δ is defined by

$$\Delta := \sum_{S_i \sim S_j} \Pr(\mathcal{B}_i \cap \mathcal{B}_j), \qquad (2.14)$$

where $S_i \sim S_j$ if the two cycles determined by S_i and S_j have at least one arc in common.

First, we find an upper bound for Δ. Letting i remain fixed, we (rather crudely) obtain

$$\Delta \leq n^s \sum_{j:S_i \sim S_j} \Pr(\mathcal{B}_i \cap \mathcal{B}_j), \tag{2.15}$$

since each $|U_r| \leq n$ and each $|S_i| = s$. The sum on the right side satisfies

$$\sum_{j:S_i \sim S_j} \Pr(\mathcal{B}_i \cap \mathcal{B}_j) \ \leq \ \sum_{r=1}^{s-1} \binom{s}{r} p^{2s-r} w^{s-(r+1)}. \tag{2.16}$$

The binomial coefficient in (2.16) accounts for the number of ways to choose the arcs of $S_i \cap S_j$, the power of p is $\Pr(\mathcal{B}_j|\mathcal{B}_i)\Pr(\mathcal{B}_i)$, and finally, the power of w reflects the facts that each U-set has cardinality w and, with i fixed, there are at most $s-(r+1)$ vertices in the S_j-cycle not already in the S_i-cycle. Recalling that $w = \lceil n/(2k')\rceil$ (so that $w < n$), using the gross bound $\binom{s}{r} < 2^s$, and replacing p with $n^{\varepsilon-1}$, we find that

$$\sum_{j:S_i \sim S_j} \Pr(\mathcal{B}_i \cap \mathcal{B}_j) \ < \ 2^s \sum_{r=1}^{s-1} p^{2s-r} n^{s-(r+1)} \ = \ 2^s \sum_{r=1}^{s-1} n^{2\varepsilon s - s - r\varepsilon - 1} \ < \ 2^s s n^{2\varepsilon s - s - \varepsilon - 1}.$$

With (2.15), the last estimate yields

$$\Delta < 2^s s n^{2\varepsilon s - \varepsilon - 1}. \tag{2.17}$$

Next, we find a lower bound for $\mu := E[X]$. Since there are ℓ U-sets, each containing w vertices, and each ordered choice of s/ℓ vertices from each (up to the choice of the first vertex) contributes 1 to X with probability at least p^s, we have

$$\mu \ \geq \ \frac{1}{s}\binom{w}{s/\ell}^{\ell} \left[\left(\frac{s}{\ell}\right)!\right]^{\ell} p^s.$$

Therefore,

$$\mu \ \geq \ \frac{1}{s}\left(\frac{w!}{(w-s/\ell)!}\right)^{\ell} p^s \ \geq \ \frac{1}{s}\left(w - \frac{s}{\ell}\right)^s p^s \ \geq \ \frac{1}{s}\left(\frac{n}{4k'}\right)^s n^{\varepsilon s - s} \ = \ \frac{n^{\varepsilon s}}{s(4k')^s}. \tag{2.18}$$

We distinguish two cases.

Case 1: $\Delta \geq \mu$.
Here, we have the hypotheses of the Extended Janson Inequality ([2, Theorem 8.1.2]), which, along with our bounds (2.17), (2.18) gives

$$\Pr(X=0) \ \leq \ e^{-\mu^2/(2\Delta)} \ < \ e^{-n^{1+\varepsilon}/(2s^3(32k'^2)^s)}.$$

Case 2: $\Delta < \mu$.
Now we have the hypotheses of the basic Janson Inequality ([2, Theorem 8.1.1]), which together with (2.18) gives

$$\Pr(X=0) \ \leq \ e^{-\mu+\Delta/2} \ < \ e^{-\mu/2} \ \leq \ e^{-n^{\varepsilon s}/(2s(4k')^s)}.$$

Let $s > 1 + (1+\varepsilon)/\varepsilon$ be a multiple of ℓ. Then the last bound shows that

$$\Pr(X=0) \;\le\; e^{-n^{1+\varepsilon}(n^{\varepsilon}/(2s(4k')^{s}))} \;\le\; e^{-n^{1+\varepsilon}}.$$

Since s and k' are constants (not depending on n), as is the number 1 (the coefficient of $n^{1+\epsilon}$ in the last expression), in either case we see that

$$P_\ell \;\le\; \Pr(X=0) \;\le\; e^{-\gamma n^{1+\varepsilon}}$$

for some constant $\gamma > 0$. This gives us Lemma 2.1. $\square$

We return to our estimation of the expected number N of bad sequences in (2.6), repeated here for convenience:

$$N \;\le\; \sum_{\ell=2}^{k} \binom{k}{\ell} (\ell-1)! \binom{n}{w}^{\ell} P_\ell.$$

Using Lemma 2.1 to bound the factors P_ℓ in this sum shows that for n large enough,

$$N \;\le\; \sum_{\ell=2}^{k} \binom{k}{\ell} (\ell-1)! \binom{n}{w}^{\ell} e^{-\gamma n^{1+\varepsilon}} \;<\; \sum_{\ell=2}^{k} \frac{e^{-n}}{2k} \;<\; \frac{e^{-n}}{2}. \tag{2.19}$$

From (2.19) and Markov's Inequality, we conclude that

$$\Pr(\exists \text{ a bad sequence}) < \frac{e^{-n}}{2}. \tag{2.20}$$

Since ϕ fails to be an acyclic homomorphism exactly when there exists a bad pair or there exists a bad sequence, (2.5) and (2.20) now show that

$$|\mathcal{D} \smallsetminus \mathcal{D}_2| \le (\Pr(\exists \text{ bad pair}) + \Pr(\exists \text{ bad sequence}))\, |\mathcal{D}| < e^{-n}\, |\mathcal{D}|\,,$$

which yields (2.2). $\square$

3. Proof of Theorem 1.5

To obtain the conclusion of Theorem 1.5 (unique D-colourability), we shall need to refine the deletion method employed in the proof of Theorem 1.2. We preserve the earlier notation. Let $\mathcal{D}_3$ be the set of digraphs $H \in \mathcal{D}_1$, in which any two cycles of length less than g are disjoint. Let $\mathcal{D}_4$ denote the set of those $H \in \mathcal{D}$ with the property that $H - A_1$ is uniquely D-colourable for any set A_1 of at most $\lceil n^{g\varepsilon} \rceil$ independent arcs. (Here, we call a set $S \subseteq E(H)$ *independent* if no two arcs in S have a vertex in common.) Now we will show that

$$|\mathcal{D}_3| > \left(1 - n^{-\varepsilon/3}\right) |\mathcal{D}| \tag{3.1}$$

and

$$|\mathcal{D}_4| > \left(1 - e^{-n^{\varepsilon}/6}\right) |\mathcal{D}|\,. \tag{3.2}$$

Since (3.1) and (3.2) imply that $\mathcal{D}_3 \cap \mathcal{D}_4 \ne \varnothing$ (for large enough n), we can choose a digraph $H \in \mathcal{D}_3 \cap \mathcal{D}_4$. As $H \in \mathcal{D}_3 \subseteq \mathcal{D}_1$, we can delete a set A_1 of at most $\lceil n^{g\varepsilon} \rceil$ independent arcs from H so that $D^* := H - A_1$ has girth at least g, and

$H \in \mathcal{D}_4$ ensures that D^* is uniquely D-colourable. Hence, to complete the proof of Theorem 1.5, it suffices to establish (3.1) and (3.2).

Proof of (3.1). For integers $\ell_1, \ell_2 < g$, we follow [31] and call a digraph an (ℓ_1, ℓ_2)-*double cycle* if it consists of a directed cycle C_{ℓ_1} of length ℓ_1 and a directed path of length ℓ_2 joining two (not necessarily distinct) vertices of C_{ℓ_1}; such a digraph contains $\ell_1+\ell_2-1$ vertices and $\ell_1+\ell_2$ arcs. Let $\mathcal{D}'$ denote the set of digraphs in $\mathcal{D}$ containing an (ℓ_1, ℓ_2)-double cycle for some $\ell_1, \ell_2 < g$. Notice that $\mathcal{D}_1 \setminus \mathcal{D}_3 \subseteq \mathcal{D}'$, whence

$$|\mathcal{D}_1 \setminus \mathcal{D}_3| \leq |\mathcal{D}'|\,, \tag{3.3}$$

so we can obtain a lower estimate for $|\mathcal{D}_3|$ by estimating $|\mathcal{D}'|$.

For fixed $\ell_1, \ell_2 < g$, the expected number $N(\ell_1, \ell_2)$ of (ℓ_1, ℓ_2)-double cycles in a digraph $H \in \mathcal{D}$ is less than

$$\ell_1 (kn)^{\ell_1} (kn)^{\ell_2 - 1} p^{\ell_1+\ell_2},$$

since there are fewer than $\ell_1 (kn)^{\ell_1} (kn)^{\ell_2-1}$ ways of choosing such a double cycle Y with $V(Y) \subseteq V$, and each such Y exists with probability 0 or $p^{\ell_1+\ell_2}$. Since $p = n^{\varepsilon-1}$ we have

$$N(\ell_1, \ell_2) < \ell_1 k^{\ell_1+\ell_2} n^{\varepsilon(\ell_1+\ell_2)} n^{-1}.$$

Since $\varepsilon(\ell_1+\ell_2) \leq 2g\varepsilon < 1/2$, for large enough n we have

$$\sum_{\substack{2 \leq \ell_1 < g \\ 1 \leq \ell_2 < g}} N(\ell_1, \ell_2) < n^{-1/2}.$$

Markov's Inequality now shows that

$$|\mathcal{D}'| < n^{-1/2}\,|\mathcal{D}|\,,$$

so from (3.3) we obtain

$$|\mathcal{D}_3| > |\mathcal{D}_1| - n^{-1/2}\,|\mathcal{D}|\,,$$

and (2.1) gives (3.1). □

Proof of (3.2). We will argue that $|\mathcal{D} \setminus \mathcal{D}_4| < e^{-n^{\varepsilon}/6} |\mathcal{D}|$. If $H \in \mathcal{D} \setminus \mathcal{D}_4$, then there is a set A_1 of at most $\lceil n^{g\varepsilon} \rceil$ independent arcs of H so that $H - A_1$ admits an acyclic homomorphism h to D that is not the composition $\sigma \circ c$ of the natural homomorphism $c\colon H - A_1 \to D$ (sending V_i to i) with an automorphism σ of D. As in the proof of (2.2), we can define a function $\phi\colon V(D) \to V(D)$ such that $\left|V_i \cap h^{-1}(\phi(i))\right| \geq n/k$ for each $i \in V(D)$.

Let us first suppose that ϕ is not an automorphism of D. By hypothesis, D is a core, so any acyclic homomorphism of D to itself must be an automorphism. It follows that ϕ is not an acyclic homomorphism. Therefore, there is an arc $ij \in E(D)$ such that $\phi(i)\phi(j) \notin E(D)$, or there is a vertex $i \in V(D)$ such that $\phi^{-1}(i)$ is not

acyclic. Notice that the current arrangement is analogous to the one in the second paragraph in the proof of (2.2). Repeating the earlier argument, with D in the place of C and k in the role of k', we find that most $H \in \mathcal{D}$ do not fall into the present case. More precisely, we reach the following conclusion:

At least $(1-e^{-n})|\mathcal{D}|$ digraphs $H \in \mathcal{D}$ have the property that for any set A_1 of at most $\lceil n^{g\varepsilon} \rceil$ arcs (independent or otherwise), the digraph $H - A_1$ cannot be D-coloured so that ϕ is not an automorphism of D.

Thus, in this case, $|\mathcal{D} \setminus \mathcal{D}_4| < e^{-n}|\mathcal{D}| < e^{-n^{\varepsilon}/6}|\mathcal{D}|$, and (3.2) is proved.

From now on, we treat the case when ϕ is an automorphism of D. Without loss of generality, we may assume that ϕ is the identity, i.e., that

$$\left|V_i \cap h^{-1}(i)\right| \geq n/k \text{ for each } i \in V(D). \tag{3.4}$$

We may assume further that

$$\left|V_j \cap h^{-1}(i)\right| < n/k \text{ for all } j \neq i. \tag{3.5}$$

(Otherwise, we can redefine $\phi(i)$ to be equal to j and fall into the case where ϕ is not an automorphism.)

Since h is not the composition $\sigma \circ c$ of the natural homomorphism $c\colon H - A_1 \to D$ (sending V_i to i) with an automorphism σ of D, there must be a pair $\{i, j\}$ of distinct vertices of D such that $V_j \cap h^{-1}(i) \neq \varnothing$. Let $\{i_0, j_0\}$ be such a pair that maximizes $|V_{j_0} \cap h^{-1}(i_0)|$. Consider the map $\phi'\colon V(D) \to V(D)$ such that

$$\phi'(x) := \begin{cases} x \ (= \phi(x)) & \text{if } x \neq j_0 \\ i_0 & \text{if } x = j_0. \end{cases}$$

Clearly ϕ' is not a bijection, and since D is a core, it cannot be an acyclic homomorphism. There are two possibilities.

Case 1: Both $j_0 i_0$ and $i_0 j_0$ are arcs of D (so $\phi'^{-1}(i_0)$ is not acyclic).

Case 2: There exists $v \in V(D)$ such that $v j_0$ is an arc of D but $v i_0$ is not, or $j_0 v$ is an arc of D but $i_0 v$ is not.

We will show that in either case, $|\mathcal{D} \setminus \mathcal{D}_4| < e^{-n^{\varepsilon}/6}|\mathcal{D}|$.

Case 1: Our choice of $\{i_0, j_0\}$ ensures that $h^{-1}(i_0) \cap V_{j_0} \neq \varnothing$. Let $x \in h^{-1}(i_0) \cap V_{j_0}$, and consider the (nonrandom) subdigraph $\widehat{D_n}$ of D_n induced by $\{x\} \cup (h^{-1}(i_0) \cap V_{i_0})$. As V_{i_0} induces no cycles, all cycles of $\widehat{D_n}$ must include x, and since the arcs of A_1 are independent, at most one such arc is incident with x. Furthermore, the constraint on the size of A_1 and our choice of ε (smaller than $1/4g$) give

$$|A_1| \leq \lceil n^{g\varepsilon} \rceil < \lceil n^{1/4} \rceil \ll \frac{n}{k}.$$

Because $|h^{-1}(i_0) \cap V_{i_0}| \geq n/k$ (cf. (3.4)), there must be a subset $U \subseteq h^{-1}(i_0) \cap V_{i_0}$ of cardinality $\lfloor n/2k \rfloor$ such that the (random) subdigraph induced by $\{x\} \cup U$ contains

no arcs of A_1 and moreover is acyclic (since $h^{-1}(i_0)$ is acyclic). To show that this is unlikely, we first estimate the expected number M of ways to select a vertex $x \in V_{j_0}$ and a subset $U \subseteq V_{i_0}$ of cardinality $\lfloor n/2k \rfloor$ so that the subdigraph $H_{x,U}$ of H that they induce is acyclic and no arc of A_1 is incident with a vertex in U. If $P_{x,U}$ denotes the probability that $H_{x,U}$ is acyclic, then

$$M \leq n\binom{n}{\lfloor n/2k \rfloor} P_{x,U} < n^n P_{x,U}. \tag{3.6}$$

In order to estimate $P_{x,U}$, we again employ the Janson Inequalities (cf. [2, Chapter 8]). Now Ω denotes the set of all potential arcs in the subdigraph $D'_{x,U}$ of D_n induced by $\{x\} \cup U$; each arc in Ω appears in $H_{x,U}$ with probability p. Let $\ell > (2+\varepsilon)/\varepsilon$ be a fixed integer. Let us index those cycles of $D'_{x,U}$ (with the positive integers) that are of length $\ell+1$ in $D'_{x,U}$. For $j \geq 1$, let S_j be the arc-set of the jth such cycle and $\mathcal{B}_j$ be the event that the arcs in S_j all appear (i.e. the cycle determined by S_j is present in $H_{x,U}$). Let X count the $\mathcal{B}_j$ that occur; since $\Pr(X=0)$ is an upper bound for $P_{x,U}$, we can bound $P_{x,U}$ by bounding $\Pr(X=0)$.

As in (2.14), Janson's Δ is given by

$$\Delta := \sum_{S_i \sim S_j} \Pr(\mathcal{B}_i \cap \mathcal{B}_j).$$

Since there are at most $\binom{\lfloor n/2k \rfloor}{\ell} < n^\ell$ cycles S_j, if we fix an S_i to maximize $\sum_{j:S_j \sim S_i} \Pr(\mathcal{B}_i \cap \mathcal{B}_j)$, then

$$\Delta \leq n^\ell \sum_{j:S_j \sim S_i} \Pr(\mathcal{B}_i \cap \mathcal{B}_j). \tag{3.7}$$

Now we sum over the number r of common arcs an S_j can have with S_i; this fixes at least $r+1$ vertices of S_j. Thus,

$$\sum_{j:S_j \sim S_i} \Pr(\mathcal{B}_i \cap \mathcal{B}_j) \;\leq\; \sum_{r=1}^{\ell} \binom{\ell+1}{r} \left\lfloor \frac{n}{2k} \right\rfloor^{\ell-r-1} p^{2(\ell+1)-r}.$$

Using the crude upper estimates $\binom{\ell+1}{r} < 2^{\ell+1}$ and $\lfloor n/2k \rfloor < n$, and replacing p with $n^{\varepsilon-1}$, we obtain

$$\sum_{j:S_j \sim S_i} \Pr(\mathcal{B}_i \cap \mathcal{B}_j) \;<\; 2^{\ell+1} \sum_{r=1}^{\ell} (np)^{\ell-r-1} p^{\ell+3} \;<\; 2^{\ell+1}\ell (np)^{\ell-2} p^{\ell+3} \;=\; 2^{\ell+1}\ell n^{2\varepsilon\ell+\varepsilon-\ell-3}.$$

This and (3.7) now give

$$\Delta \leq 2^{\ell+1}\ell n^{2\varepsilon\ell+\varepsilon-3}. \tag{3.8}$$

We also need to find a lower bound for $\mu := E[X]$. Since the arcs of $D'_{x,U}$ within U are acyclically oriented, each choice of ℓ vertices within U determines exactly one

potential $(\ell+1)$-cycle (viz., through x). It follows that

$$\mu \;=\; \binom{\lfloor n/2k\rfloor}{\ell}p^{\ell+1} \;>\; \left(\frac{\lfloor n/2k\rfloor}{\ell}\right)^{\ell}p^{\ell+1} \;>\; \frac{n^{\varepsilon\ell+\varepsilon-1}}{(4k\ell)^{\ell}}. \tag{3.9}$$

As in the proof of Theorem 1.2, we have two subcases.

Subcase 1(i): $\Delta \geq \mu$.

Again, we have the hypotheses of the Extended Janson Inequality ([2, Theorem 8.1.2]), which, along with (3.8) and (3.9) gives

$$\Pr(X=0) \;\leq\; e^{-\mu^2/(2\Delta)} \;<\; e^{-n^{1+\varepsilon}/(\ell 2^{\ell+2}(4k\ell)^{2\ell})} =: e^{-\beta n^{1+\varepsilon}},$$

where β is the (positive) constant (not depending on n) absorbing the denominator in the preceding exponent.

Subcase 1(ii): $\Delta < \mu$.

Here, we have the hypotheses of the Janson Inequality ([2, Theorem 8.1.1]), which, with the help of (3.9) gives

$$\Pr(X=0) \;\leq\; e^{-\mu+\Delta/2} \;<\; e^{-\mu/2} \;<\; e^{-n^{\varepsilon\ell+\varepsilon-1}/(2(4k\ell)^{\ell})}.$$

Recalling our choice of $\ell > (2+\varepsilon)/\varepsilon$, we see that

$$\Pr(X=0) \;<\; e^{-n^{1+2\varepsilon}/(2(4k\ell)^{\ell})} \;<\; e^{-n^{1+\varepsilon}}.$$

In either subcase, we have that $P_{x,U} \leq \Pr(X=0) < e^{-\beta n^{1+\varepsilon}}$ (since $\beta < 1$), and returning to (3.6), we have

$$M \;<\; n^n P_{x,U} \;<\; n^n e^{-\beta n^{1+\varepsilon}} \;=\; \left(ne^{-\beta n^{\varepsilon}}\right)^n \;<\; e^{-\beta n^{1+\varepsilon}/2}.$$

By Markov's Inequality, the probability that there exists such an $\{x\}\cup U$ (that induces an acyclic subdigraph) is less than $e^{-\beta n^{1+\varepsilon}/2} < e^{-n^{\varepsilon}/6}$, and so in Case 1, $|\mathcal{D}\smallsetminus\mathcal{D}_4| < e^{-n^{\varepsilon}/6}|\mathcal{D}|$, as desired.

Case 2: By the hypothesis of this case, there is a vertex v such that either $vj_0 \in E(D)$ and $vi_0 \notin E(D)$, or $j_0v \in E(D)$ and $i_0v \notin E(D)$. We will consider the first of these; the second one yields to similar reasoning. Let us recall that we chose a pair $\{i_0, j_0\}$ of distinct vertices of D so as to maximize $b := |V_{j_0}\cap h^{-1}(i_0)| \neq 0$.

Claim: Every vertex $z \in V(D)\smallsetminus\{i_0\}$ satisfies $\left|V_z\cap h^{-1}(z)\right| \geq n-(k-1)b$.

Proof of claim. Otherwise, some $z \neq i_0$ satisfies $\left|V_z\cap h^{-1}(z)\right| < n-(k-1)b$. By the pigeonhole principle, there is some $u \neq z$ such that $\left|V_z\cap h^{-1}(u)\right| > b$, but this contradicts our choice of $\{i_0, j_0\}$. □

Using the claim, we see that there are sets $U_v \subseteq V_v\cap h^{-1}(v)$ and $U_{j_0} = V_{j_0}\cap h^{-1}(i_0)$ with $|U_v| = n-(k-1)b$ and $|U_{j_0}| = b$. Since $h\colon H-A_1\to D$ is an acyclic homomorphism and $vi_0\notin E(D)$, there are at most $\min\{b, \lceil n^{g\varepsilon}\rceil\}$ independent arcs from a vertex in U_v to one in U_{j_0}. We now estimate the expected number $L(b)$ of

pairs $U'_v \subseteq V_v$, $U'_{j_0} \subseteq V_{j_0}$ with $|U'_v| = n - (k-1)b = n - (k-1)|U'_{j_0}|$, and at most $\min\{b, \lceil n^{g\varepsilon} \rceil\}$ arcs from U'_v to U'_{j_0}.

For $b < n/k$ (cf. (3.5)) and $s \leq \min\{b, \lceil n^{g\varepsilon} \rceil\}$, denote by $L(b,s)$ the expected number of pairs $U'_v \subseteq V_v$, $U'_{j_0} \subseteq V_{j_0}$, $|U'_v| = n - (k-1)b = n - (k-1)|U'_{j_0}|$, and exactly s arcs joining a vertex in U'_v to one in U'_{j_0}. Then

$$\begin{aligned} L(b,s) &< \binom{n}{n-(k-1)b}\binom{n}{b}\binom{(n-(k-1)b)b}{s} p^s (1-p)^{(n-(k-1)b)b-s} \\ &< n^{(k-1)b} n^b (nb)^s n^{s(\varepsilon-1)} e^{-bn^\varepsilon + n^{\varepsilon-1}((k-1)b^2+s)} \\ &< b^s n^{\varepsilon s} n^{kb} e^{-(bn^\varepsilon)/2} \\ &= b^s n^{\varepsilon s} (n^k e^{-n^\varepsilon/2})^b \\ &< b^s n^{\varepsilon s} e^{-(bn^\varepsilon)/3} \\ &< e^{-n^\varepsilon/4}. \end{aligned}$$

Letting $L(b) = \sum_{s \leq \min\{b, \lceil n^{g\varepsilon} \rceil\}} L(b,s) < \lceil n^{g\varepsilon} \rceil e^{-n^\varepsilon/4} < e^{-n^\varepsilon/5}$, we find that

$$\sum_{1 \leq b < n/k} L(b) < (n/k) e^{-n^\varepsilon/5} < e^{-n^\varepsilon/6}.$$

This completes the discussion for the case when $vj_0 \in E(D)$ and $vi_0 \notin E(D)$; an identical argument gives the same upper bound in the case when $j_0 v \in E(D)$ and $i_0 v \notin E(D)$. Thus in Case 2, we also arrive at $|\mathcal{D} \smallsetminus \mathcal{D}_4| < e^{-n^\varepsilon/6}|\mathcal{D}|$.

Combining the estimates obtained above and applying Markov's Inequality finally yields (3.2) and therefore completes the proof of Theorem 1.5. $\square$

4. The circular chromatic number

We turn now to the implications of Theorem 1.5 for circular colouring digraphs. The concept of the digraph circular chromatic number χ_c, defined below, generalizes the circular chromatic number for undirected graphs. The theory of the graph invariant, as of 2001, was surveyed in [33]. The digraph version was introduced in [4], where it was proved, via Lemma 4.2 below, that χ_c assumes all rational values at least one. (Note that the digraphs of Lemma 4.2 do not generally have large girth.) The same article also established the following analogue of the Erdős' theorem introducing the present paper: there exist digraphs with arbitrarily large girth and arbitrarily large circular chromatic number (this is the result to which we alluded immediately following the statement of Theorem 1.2). Our main result here, Theorem 4.4, provides a common generalization and strengthening of these basic results. It shows that the 'all conceivable rationals' property of χ_c holds even for digraphs of arbitrarily large girth and even demanding a certain uniqueness of the colouring.

Let $d \geq 1$ and $k \geq d$ be integers. Let $C(k,d)$ be the digraph with vertex set $\mathbb{Z}_k = \{0,1,\ldots,k-1\}$ and arcs

$$E(C(k,d)) = \{ij \mid j - i \in \{d, d+1, \ldots, k-1\}\},$$

where the subtraction is considered in the cyclic group $\mathbb{Z}_k$ of integers modulo k.

Acyclic homomorphisms into $C(k,d)$ are an important concept because of their relation to the circular chromatic number of digraphs; cf. [4]. An acyclic homomorphism of a digraph D into $C(k,d)$ is called a (k,d)-*colouring* of D. It is shown in [4, 21] that there is a rational number $q \in \mathbb{Q}$ such that D has a (k,d)-colouring if and only if $k/d \geq q$. This value q is denoted by $\chi_c(D)$ and called the *circular chromatic number* of D. For $q \in \mathbb{Q}^+$, let S_q denote the circle of perimeter q (centred, say, at the origin of $\mathbb{R}^2$). We define a *circular q-colouring* of D to be a map $\phi : V(D) \to S_q$ such that for every $xy \in E(D)$, with $\phi(x) \neq \phi(y)$, the distance $d_S(\phi(x), \phi(y))$ from $\phi(x)$ to $\phi(y)$ in the clockwise direction around S_q is at least 1, and for every $p \in S_q$, the preimage $\phi^{-1}(p)$ induces an acyclic subdigraph of D. If ϕ is a circular q-colouring, we say that the arc $xy \in E(D)$ is *tight* whenever $d_S(\phi(x), \phi(y)) \leq 1$ (in which case this distance is either 1 or 0). A cycle in D consisting of tight arcs is called a *tight cycle* for the circular q-colouring ϕ. Note that every tight cycle contains an arc xy such that $d_S(\phi(x), \phi(y)) = 1$. We will use the following results, respectively from [21] and [4].

Lemma 4.1. *If $\chi_c(D) = q$, then every circular q-colouring of D has a tight cycle.*

Lemma 4.2. $\chi_c(C(k,d)) = k/d$.

Lemmas 4.1 and 4.2 imply the following fact.

Proposition 4.3. *If k and d are integers with $1 \leq d \leq k$, then $C(k,d)$ is a core if and only if k and d are relatively prime.*

Proof. Let $C = C(k,d)$ and $V = V(C)$. If $r := \gcd(k,d) > 1$, then the mapping $\phi\colon V \to V$ given by $\phi(i) := r\lfloor i/r \rfloor$ is easily seen to be an acyclic homomorphism $C \to C$ that is not surjective. By Lemma 1.3, $C(k,d)$ is not a core.

For the converse, assume that k and d are relatively prime, and let $\phi\colon V \to V$ be an acyclic homomorphism. Define a map $\varphi\colon C(k,d) \to S_{k/d}$ as follows. Let $s_0, s_1, \ldots, s_{k-1}$ be points on $S_{k/d}$ that appear on the circle consecutively at distance $1/d$ apart. For $0 \leq i \leq k-1$, we set $\varphi(i) := s_{\phi(i)}$. Since ϕ is an acyclic homomorphism, it is easily verified that φ is a circular $\frac{k}{d}$-colouring of $C(k,d)$. By Lemmas 4.1 and 4.2, φ has a tight cycle $C_0 = v_1 v_2 \cdots v_\ell v_1$ in $C(k,d)$. We may assume that $\varphi(v_1) = s_0$. The images $\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_\ell), \varphi(v_1)$ must take consecutive values $s_0, s_d, s_{2d}, s_{3d}, \ldots$ (each possibly repeated several times), with the indices taken modulo k, and end up at s_0. Since k and d are relatively prime, the

sequence $s_0, s_d, s_{2d}, \ldots$ must exhaust all the elements in the set $\{s_0, s_1, \ldots, s_{k-1}\}$. This shows that ϕ is surjective; whence, by Lemma 1.3, $C(k, d)$ is a core. □

Proposition 4.3 and Theorem 1.5 together yield an immediate consequence, Corollary 1.6, that we now state in a slightly expanded (and more precise) form:

Theorem 4.4. *If k and d are relatively prime integers with $1 \leq d \leq k$, then for every positive integer g, there exists a uniquely $C(k, d)$-colourable digraph of girth at least g (and with circular chromatic number equal to k/d).*

The last claim of Theorem 4.4 follows from the next result, an analogue of [31, Theorem 3].

Theorem 4.5. *If D is a uniquely $C(k, d)$-colourable digraph, then $\chi_{\mathrm{c}}(D) = k/d$.*

Proof. Since D is $C(k, d)$-colourable, we have $\chi_{\mathrm{c}}(D) \leq k/d$. Suppose, for a contradiction, that $\chi_{\mathrm{c}}(D) = k'/d' < k/d$. Define $d^* := dd'$, $m := kd'$ and $m' := k'd$ so that $m'/d^* = k'/d' < k/d = m/d^*$. Now, let ϕ' be an (m', d^*)-colouring of D. Using the idea in the proof of Proposition 4.3, we can define a circular $\frac{m'}{d^*}$-colouring φ of $C(m', d^*)$ so that $\varphi \circ \phi'$ is such a colouring of D. Since $\chi_{\mathrm{c}}(D) = k'/d' = m'/d^*$, Lemma 4.1 implies that $\varphi \circ \phi'$ has a tight cycle in D. Choosing a tight arc $xy \in E(D)$ for $\varphi \circ \phi'$ yields an arc xy of D such that $\phi'(x)$ and $\phi'(y)$ are separated by d^* units in the clockwise direction around $C(m', d^*)$. Without loss of generality, we may assume that $\phi'(y) = 0$ and $\phi'(x) = m' - d^*$. We define an (m, d^*)-colouring ψ as follows: $\psi(v) := \phi'(v)$ if $\phi'(v) < m' - d^*$ and $\psi(v) := \phi'(v) + m - m'$ otherwise. It is easily verified that ψ is indeed an (m, d^*)-colouring of D. Next, we define $\bar{\psi}\colon V(D) \to V(C(k, d))$ by $\bar{\psi}(v) := \lfloor \psi(v)/d' \rfloor$. (In this formula—and hereafter—we view the vertices $\psi(v)$ of $C(m, d^*)$ as integers between 0 and $m - 1$.) Since $\lfloor \cdot/d' \rfloor\colon V(C(m, d^*)) \to V(C(k, d))$ defines an acyclic homomorphism, and such maps compose (cf. [4]), it is not hard to check that $\bar{\psi}$ is a (k, d)-colouring of D. Similarly, we define $\bar{\phi}\colon V(D) \to V(C(k, d))$ by $\bar{\phi}(v) := \lfloor \phi'(v)/d' \rfloor$. As in the case of $\bar{\psi}$, it is easy to check that $\bar{\phi}$ is a (k, d)-colouring of D. We claim that $\bar{\phi}$ and $\bar{\psi}$ do not differ by an automorphism of $C(k, d)$. Note that $\bar{\phi}(y) = \bar{\psi}(y) = 0$; therefore, it suffices to show that $\bar{\phi}(x) \neq \bar{\psi}(x)$. Now,

$$\bar{\phi}(x) = \left\lfloor \frac{m' - d^*}{d'} \right\rfloor = \left\lfloor \frac{d(k' - d')}{d'} \right\rfloor \qquad \text{while} \qquad \bar{\psi}(x) = \left\lfloor \frac{m - d^*}{d'} \right\rfloor = k - d.$$

Since $k'/d' < k/d$, we have $d(k' - d')/d' < k - d$, and it follows that $\bar{\phi}(x) < \bar{\psi}(x)$. This implies that $\bar{\phi}$ and $\bar{\psi}$ are (k, d)-colourings of D that do not differ by an automorphism of $C(k, d)$. Hence, D is not uniquely $C(k, d)$-colourable, a contradiction. □

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